Aula 6

Teorema Slide → Prova dos pontos (ii) e (iii)

Seja $f(x) = \sum_{n=0}^{+\infty} a_n (x-c)^n$. Pelos teoremas anteriores, a série converge uniformemente e por isso podemos aplicar o Teorema do Slide 23.

(ii)

$$f'(x) = \left(\sum_{n=0}^{+\infty} a_n (x-c)^n\right)' \underset{\text{Teo. slide 23(iii)}}{=} \sum_{n=1}^{+\infty} \left(a_n (x-c)^n\right)' = \sum_{n=1}^{+\infty} n a_n (x-c)^{n-1}$$

(iii) Nota: Uma primitiva de f tal que $F(c)=0$: $F(x) = \int_c^x f(t)\,dt$

$$F(x) = \int_c^x f(t)\,dt = \int_c^x \left(\sum_{n=0}^{+\infty} a_n (t-c)^n\right) dt \underset{\text{Teo. slide 23}}{=} \sum_{n=0}^{+\infty} \int_c^x a_n (t-c)^n\,dt$$

$$= \sum_{n=0}^{+\infty} \left[a_n \times \frac{(t-c)^{n+1}}{n+1}\right]_c^x = \sum_{n=0}^{+\infty} \left(a_n \times \frac{(x-c)^{n+1}}{n+1} - \underbrace{a_n \times \frac{(c-c)^{n+1}}{n+1}}_{=0}\right)$$

$$= \sum_{n=0}^{+\infty} a_n \times \frac{(x-c)^{n+1}}{n+1} \checkmark$$

Exercício 1. Recordar Formulário → $\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n, \; -1 < x < 1$

a)

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2} \times \frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{1}{2} \times \sum_{n=0}^{+\infty} \left(-\frac{x}{2}\right)^n, \; -1 < -\frac{x}{2} < 1$$

$$= \frac{1}{2} \times \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{2^n}, \; -1 < \frac{x}{2} < 1$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{2^{n+1}}, \; -2 < x < 2$$

b)

$$\frac{1}{x} = \frac{1}{1-1+x} = \frac{1}{1-(1-x)} \overset{\text{formulário}}{=} \sum_{n=0}^{+\infty} (1-x)^n, \; -1 < 1-x < 1$$

$$= \sum_{n=0}^{+\infty} (-(x-1))^n, \; -2 < -x < 0$$

$$= \sum_{n=0}^{+\infty} (-1)^n (x-1)^n, \; 0 < x < 2$$

c) $\dfrac{1}{(1-x)^2}$ Nota $\left(\dfrac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$, $\forall x \in]-1,1[$

Logo, pelo Teo. Slide 30 (ii)

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{+\infty} x^n\right)' = \sum_{n=1}^{+\infty} (x^n)' = \sum_{n=1}^{+\infty} n\, x^{n-1}, -1<x<1$$

formulário

ou, usando $K = n-1$ $\Leftrightarrow K+1=n$

$$= \sum_{K=0}^{+\infty} (K+1)x^K, -1<x<1$$

d) $\dfrac{2}{(1-x)^3}$ (usar alínea c)) Nota: $\left(\dfrac{1}{(1-x)^2}\right)' = \dfrac{2}{(1-x)^3}$

Logo, pelo Teo. Slide 30 (ii)

$$\frac{2}{(1-x)^3} = \left(\frac{1}{(1-x)^2}\right)' \overset{\text{alínea c}}{=} \left(\sum_{n=1}^{+\infty} n x^{n-1}\right)' = \sum_{n=2}^{+\infty} (n x^{n-1})'$$

$$= \sum_{n=2}^{+\infty} n \times (n-1) x^{n-2}, -1<x<1$$

e) $-\ln(1-x)$ Nota $\int \frac{u'}{u}\,dx = \ln|u| + c, c \in \mathbb{R}$

Seja $f(x) = \dfrac{1}{1-x}$, $-1<x<1$

A primitiva de f tal que $F(0)=0$ é $F(x) = \int_0^x \frac{1}{1-t}\,dt = -\left[\ln|1-t|\right]_0^x$

$= -\left(\ln(1-x) - \underbrace{\ln(1-0)}_{=0}\right)$

$= -\ln(1-x)$

$u = 1-t$

$u' = (1-t)' = -1$

Como $f(x) = \dfrac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$, $-1<x<1$

pelo Teo. Slide 30 (iii) temos que

$$F(x) = -\ln(1-x) = \sum_{n=0}^{+\infty} \frac{1}{n+1} x^{n+1}, -1<x<1$$

f) $\operatorname{arctg} x$ Nota: $\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + K, K \in \mathbb{R}$

Seja $f(x) = \dfrac{1}{1+x^2}$ A primitiva de f tal que $F(0)=0$ é $F(x) = \int_0^x \frac{1}{1+t^2}\,dt$

$= \left[\operatorname{arctg} t\right]_0^x$

$= \operatorname{arctg} x - 0$

$= \operatorname{arctg} x$

(c. aux ⟶) desenvolvimento em série de $\dfrac{1}{1+x^2}$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{+\infty} (-x^2)^n, -1<-x^2<1 = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$$

$\boxed{-1<x^2<1}$

$\underbrace{x^2>-1}_{\text{cond. univ.}} \wedge \underbrace{x^2<1}$

$-1<x<1$

Como $f(x) = \dfrac{1}{1+x^2} = \sum_{n=1}^{+\infty} (-1)^n x^{2n}, -1<x<1$, pelo Teo. Slide 30 (iii)

Temos que $F(x) = \operatorname{arctg} x = \sum_{n=1}^{+\infty} \dfrac{(-1)^n}{2n+1} x^{2n+1}, -1<x<1$

Nota: $\int x^{2n}\,dx = \dfrac{x^{2n+1}}{2n+1} + c, c \in \mathbb{R}$

Exercício extra:

g) $\ln(2+x)$ ⇝ usar a alínea

Seja $f(x)=\dfrac{1}{2+x}$, $-2<x<2$

A primitiva de f tal que $F(0)=0$ é

$$F(x)=\int_0^x \frac{1}{2+t}\,dt=\left[\ln|2+t|\right]_0^x=\ln(2+x)-\ln(2)$$

$u=2+t$

$u'=1$

Como $f(x)=\dfrac{1}{2+x}=\displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n x^n}{2^{n+1}}$, $-2<x<2$, então pelo Teo. Slide 30(iii)

alínea a

$$F(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+1}}{(2^{n+1})(n+1)},\quad -2<x<2$$

$\ln(2+x)-\ln(2)$

Logo $\ln(2+x)=\ln(2)+\displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+1}}{2^{n+1}(n+1)}$, $-2<x<2$

Exercício 2: (Ex. 3 → F. f. 2)

função soma

a) $1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots+(-1)^n\dfrac{x^n}{n!}+\ldots=\displaystyle\sum_{n=0}^{+\infty}(-1)^n\frac{x^n}{n!}=\sum_{n=0}^{+\infty}\frac{(-x)^n}{n!}=e^{-x}$

formulário

$x\in\mathbb{R}$

b) $1-x^3+x^6-x^9+\ldots+(-1)^n x^{3n}+\ldots=\displaystyle\sum_{n=0}^{+\infty}(-1)^n x^{3n}=\sum_{n=0}^{+\infty}(-x^3)^n=\frac{1}{1-(-x^3)}$, $-1<-x^3<1$

$=\dfrac{1}{1+x^3}$, $-1<x<1$

Exercício 3: (Ex. 9 → F.P. 2)

a) $e^x=\displaystyle\sum_{n=0}^{+\infty}\frac{x^n}{n!}$, $\forall x\in\mathbb{R}$

Desenvolvimento em série de potências de xe^{x^3}

Logo, $e^{x^3}=\displaystyle\sum_{n=0}^{+\infty}\frac{(x^3)^n}{n!}=\sum_{n=0}^{+\infty}\frac{x^{3n}}{n!}$, $\forall x\in\mathbb{R}$ ⇝ $xe^{x^3}=x\times\displaystyle\sum_{n=0}^{+\infty}\frac{x^{3n}}{n!}=$

$=\displaystyle\sum_{n=0}^{+\infty}\frac{x^{3n+1}}{n!}$, $\forall x\in\mathbb{R}$

b) Calcular a representação da série numérica de $\int_0^1 xe^{x^3}dx$

$$\int_0^1 xe^{x^3}dx\underset{\text{alínea a)}}{=}\int_0^1\left(\sum_{n=0}^{+\infty}\frac{x^{3n+1}}{n!}\right)dx\underset{\text{Teo. slide 30}}{=}\sum_{n=0}^{+\infty}\left[\frac{1}{n!}\times\frac{x^{3n+2}}{3n+2}\right]_0^1=\sum_{n=0}^{+\infty}\left(\frac{1}{n!}\times\frac{1^{3n+2}}{3n+2}-0\right)$$

$$=\sum_{n=0}^{+\infty}\frac{1}{n!(3n+2)}$$

Exercício 4:

$$f(x) = \sum_{n=0}^{+\infty} \underbrace{\frac{(-1)^n \, 2^{2n+1}}{2n+1}}_{a_{2n+1}} x^{2n+1}$$

a) Calcular $f^{(31)}(0)$

$2n+1 = 31 \Leftrightarrow \boxed{n = 15}$

Nota: Pelo Teorema do slide 31 tem-se que

$\dfrac{f^{(31)}(0)}{31!}$ é igual ao coeficiente de x^{31} na série

$$\frac{(-1)^{15} \times 2^{2\times 15+1}}{2\times 15+1} = -\frac{2^{31}}{31}$$

Logo $\dfrac{f^{(31)}(0)}{31!} = \dfrac{-2^{31}}{31} \Leftrightarrow f^{(31)}(0) = \dfrac{-2^{31} \times 31!}{31} = \dfrac{-2^{31} \times 31 \times 30!}{31} = -2^{31} \times 30!$

b) Calcular $f^{(1000)}(0)$

$2n+1 = 1000 \Leftrightarrow n = \dfrac{999}{2} \notin \mathbb{N}_0 \rightsquigarrow$ Não existe termo em x^{1000} na série

Coeficiente de x^{1000} é zero

Logo $\dfrac{f^{(1000)}(0)}{1000!} = 0 \Leftrightarrow f^{(1000)}(0) = 0$